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V. *An Appendix to the Paper in the Philosophical Transactions for the Year 1778, Number XLII, pages 902 et seq. intituled, "A Method of extending Cardan's Rule for resolving one Case of the Cubick Equation $x^3 - qx = r$ to the other Case of the same Equation, which it is not naturally fitted to solve, and which is therefore called the irreducible Case."* By Francis Maferes, Esq. F. R. S. *Cursitor Baron of the Exchequer.*

Read Nov. 4, 1779.

ARTICLE I.

IN the above-mentioned paper in the Philosophical Transactions the expression $\sqrt[3]{e}x$ the infinite series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ is shewn to be equal to the root of the equation $x^3 - qx = r$, whenever $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, but greater than one half of it, or than $\frac{q^3}{54}$. This expression is wholly transcendental, or composed of an infinite number of terms, to wit, the terms of the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ multiplied into the cube-root of e . But I have since thought that it might be convenient on some occasions to divide this expression, if possible, into two others, whereof the one should be a

mere algebraick expreffion, or confift of a finite number of terms, and the other fhould be tranfcendental, or involve in it an infinite feries. And I have accordingly difcovered a method of doing this, which I will now proceed to defcribe.

ART. 2. In the above-mentioned paper in the Philofophical Tranfactions I denoted the excefs of $\frac{q^3}{27}$ above $\frac{rr}{4}$ in the fecond cafe of the equation $x^3 - qx = r$, as well as the excefs of $\frac{rr}{4}$ above $\frac{q^3}{27}$ in the firft cafe of it, by the letters ss . But I have fince thought that it might have been better to denote the excefs of $\frac{q^3}{27}$ above $\frac{rr}{4}$ in the fecond cafe of that equation by the letters zz , in order the more clearly to diftinguifh it from the oppofite difference $\frac{rr}{4} - \frac{q^3}{27}$ in the firft cafe of it, which was denoted by ss . And I therefore in the courfe of the following pages fhall ufe the letters zz inftead of ss to denote the faid excefs of $\frac{q^3}{27}$ above $\frac{rr}{4}$ in the fecond cafe of the faid equation, or the difference $\frac{q^3}{27} - \frac{rr}{4}$.

ART. 3. Now, if zz be fubftituted inftead of ss in the expreffion $\sqrt[3]{e} \times$ the infinite feries $2 + \frac{2ss}{9e} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - 8xc$. that expreffion will thereby be converted into the following expreffion, to wit, $\sqrt[3]{e} \times$ the infinite feries $2 + \frac{2zz}{9e} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} - 8xc$. Therefore, if $\frac{rr}{4}$ be
lefs

less than $\frac{q^3}{27}$ but greater than $\frac{q^3}{54}$, and e be put $= \frac{r}{2}$, and xx be put $= \frac{q^3}{27} - \frac{rr}{4}$, the root of the equation $x^3 - qx = r$ will be equal to $\sqrt[3]{e} \times$ the infinite series $2 + \frac{2xz}{9e} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} - \&c.$

ART. 4. The numeral coefficients $\frac{2}{9}$, $\frac{20}{243}$, $\frac{308}{6561}$, &c. of $\frac{2z}{e}$, $\frac{z^4}{e^4}$, $\frac{z^6}{e^6}$, &c. in this series are exactly double of $\frac{1}{9}$, $\frac{10}{243}$, $\frac{154}{6561}$, &c. which are the numeral coefficients of the same powers of the fraction $\frac{z}{e}$ in the series $1 + \frac{z}{3e} - \frac{2z^3}{9e^3} + \frac{5z^5}{81e^5} - \frac{10z^7}{243e^7} + \frac{22z^9}{729e^9} - \frac{154z^{11}}{6561e^{11}} + \frac{2618z^{13}}{1377781e^{13}} - \&c.$ which is equal to the cube-root of the binomial quantity $1 + \frac{z}{e}$; or, if the numeral coefficients of the said latter series be denoted by the capital letters A, B, C, D, E, F, G, H, &c. respectively, so that A shall be $= 1$, $B = \frac{1}{3}$, $C = \frac{1}{9}$, $D = \frac{5}{81}$, $E = \frac{10}{243}$, $F = \frac{22}{729}$, $G = \frac{154}{6561}$, and $H = \frac{2618}{1377781}$, and so on, the said numeral coefficients $\frac{2}{9}$, $\frac{20}{243}$, $\frac{308}{6561}$, &c. will be equal to $2C$, $2E$, $2G$, &c. and the series mentioned in the last Article will be $2A + \frac{2Cz^3}{e^3} - \frac{2Ez^5}{e^5} + \frac{2Gz^7}{e^7} - \&c.$ and consequently the root of the equation $x^3 - qx = r$, in the second case of it, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, will be equal to the expression $\sqrt[3]{e} \times$ the series $2A + \frac{2Cz^3}{e^3} - \frac{2Ez^5}{e^5} + \frac{2Gz^7}{e^7} - \&c.$

ART. 5. Now the series $2A + \frac{2Cz^3}{e^3} - \frac{2Ez^5}{e^5} + \frac{2Gz^7}{e^7} - \&c.$

is

is equal to the sum of the two following serieses, to wit,

$2A - \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} - \frac{2Gz^6}{e^6} - \&c.$ (in which all the terms following the first term are marked with the sign $-$, or are subtracted from the first term), and $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} + \&c$;

and the series $2A - \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} - \frac{2Gz^6}{e^6} - \&c.$ is equal to

the sum of the two serieses $A + \frac{Bz}{e} - \frac{Czz}{ee} + \frac{Dz^3}{e^3} - \frac{Ez^4}{e^4} + \frac{Fz^5}{e^5} - \frac{Gz^6}{e^6} + \&c.$ and $A - \frac{Bz}{e} - \frac{Czz}{ee} - \frac{Dz^3}{e^3} - \frac{Ez^4}{e^4} - \frac{Fz^5}{e^5} - \frac{Gz^6}{e^6} - \&c.$ which are respectively equal to the cube-roots of the

binomial quantities $1 + \frac{z}{e}$ and $1 - \frac{z}{e}$. Therefore the

series $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} - \&c.$ is $= \sqrt[3]{1 + \frac{z}{e}} + \sqrt[3]{1 - \frac{z}{e}}$ +

the infinite series $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} + \&c.$ Consequently the

expression $\sqrt[3]{e} \times$ the series $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} - \&c.$

is $= \sqrt[3]{e} \times \sqrt[3]{1 + \frac{z}{e}} + \sqrt[3]{e} \times \sqrt[3]{1 - \frac{z}{e}} + \sqrt[3]{e} \times$ the infinite

series $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} + \&c. = \sqrt[3]{e+z} + \sqrt[3]{e-z} + \sqrt[3]{e} \times$ the

infinite series $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} + \&c. = \sqrt[3]{e+z} + \sqrt[3]{e-z} +$

$4\sqrt[3]{e} \times$ the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} + \&c.$ There-

fore the root of the equation $x^3 - qx = r$, in the second case of it, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, is equal to

$\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} + \&c$; of which expression the first part, to

wit,

wit, $\sqrt[3]{e+z} + \sqrt[3]{e-z}$ is algebraick, and the latter part, to wit, $\sqrt[3]{e} \times$ the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} + \&c.$ is transcendental. Q. E. I. ^(a)

Of the convergency of the Series obtained in the preceding Article.

ART. 6. This series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} + \&c.$ evidently converges faster than the series $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} - \&c.$ or $2 + \frac{2zz}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} - \&c.$: and consequently the expression $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} + \&c.$ seems rather fitter

(a) N. B. I have been informed that both this mixed expression of the root of the equation $x^3 - qx = r$ in the second case of it, and the merely transcendental expression of it published in the former paper, and from which this expression is derived, were invented by Monsieur NICOLE, and published in the memoirs of the French Academy of Sciences so long ago as the year 1738; and the latter of them, to wit, the transcendental expression $\sqrt[3]{e} \times$ the series $2 + \frac{2z}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} - \&c.$ I had myself seen many years ago in Monsieur CLAIRAUT's algebra, in the place cited in the 50th Article of my former paper, to wit, in pages 286, 287, 288. But it was obtained by the intervention of negative quantities, and the roots of negative quantities, which gave it, in my opinion, an air of great obscurity. And therefore I thought an investigation of the same series, by a method that keeps clear of those difficulties, might not be unacceptable to the lovers of these sciences, nor unworthy of a place in the Transactions of this learned body.

to exhibit the value of x in the equation $x^3 - qx = r$, in the second case of it (in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$), to a considerable degree of exactness than the former expression $\sqrt[3]{e} \times$ the series $2 + \frac{2xz}{9ee} - \frac{20x^2}{243e^3} + \frac{308x^3}{6561e^5} - \&c.$

A Computation of the four first Terms of the Series obtained in Art. 5.

ART. 7. The first fifteen terms of the infinite series which is equal to the cube-root of $1 + \frac{z}{e}$ are as follows ;

$$\text{to wit, } 1 + \frac{z}{3e} - \frac{z^2}{9ee} + \frac{5z^3}{81e^3} - \frac{10z^4}{243e^4} + \frac{22z^5}{729e^5} - \frac{154z^6}{6561e^6} + \frac{2618z^7}{137781e^7} - \frac{935z^8}{59049e^8} + \\ \frac{21505z^9}{1594323e^9} - \frac{55913z^{10}}{4782969e^{10}} + \frac{147407z^{11}}{14348907e^{11}} - \frac{1179256z^{12}}{129140163e^{12}} + \frac{3174920z^{13}}{387420489e^{13}} - \\ \frac{60323480z^{14}}{8135830269e^{14}} + \&c; \text{ or, in decimal fractions, } 1 +$$

$$\begin{aligned} &.333,333,333, \&c. \times \frac{z}{e} - .111,111,111, \&c. \times \frac{z^2}{ee} + \\ &.061,728,395, \&c. \times \frac{z^3}{e^3} - .041,152,263, \&c. \times \frac{z^4}{e^4} + \\ &.030,178,326, \&c. \times \frac{z^5}{e^5} - .023,472,031, \&c. \times \frac{z^6}{e^6} + \\ &.019,001,167, \&c. \times \frac{z^7}{e^7} - .015,834,305, \&c. \times \frac{z^8}{e^8} + \\ &.013,488,482, \&c. \times \frac{z^9}{e^9} - .011,690,017, \&c. \times \frac{z^{10}}{e^{10}} + \\ &.010,273,045, \&c. \times \frac{z^{11}}{e^{11}} - .009,131,595, \&c. \times \frac{z^{12}}{e^{12}} + \\ &.008,195,021, \&c. \times \frac{z^{13}}{e^{13}} - .007,414,542, \&c. \times \frac{z^{14}}{e^{14}}. \end{aligned}$$

Therefore the four first terms of the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} +$

$\frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} + \&c.$ are $\frac{zz}{9ee} + \frac{154z^6}{6561e^6} + \frac{55913z^{10}}{4,782,969e^{10}} + \frac{60,323,480z^{14}}{8,135,830,269e^{14}},$
 or, in decimal fractions, .111,111,111, &c. $\times \frac{zz}{ee} +$
 .023,472,031, &c. $\times \frac{z^6}{e^6} + .011,690,017, \&c. \times \frac{z^{10}}{e^{10}} +$
 .007,414,542, &c. $\times \frac{z^{14}}{e^{14}}.$ Therefore the root of the
 cubick equation $x^3 - qx = r$, in the second case of it (in
 which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$), is equal to $\sqrt[3]{e+z} + \sqrt[3]{e-z} +$
 $4\sqrt[3]{e} \times$ the series $\frac{zz}{9ee} + \frac{154z^6}{6561e^6} + \frac{55,913z^{10}}{4,782,969e^{10}} + \frac{60,323,480z^{14}}{8,135,830,269e^{14}} + \&c.$
ad infinitum, or $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the series
 .111,111,111, &c. $\times \frac{zz}{ee} + .023,472,031, \&c. \times \frac{z^6}{e^6} +$
 .011,690,017, &c. $\times \frac{z^{10}}{e^{10}} + .007,414,542, \&c. \times \frac{z^{14}}{e^{14}} +$
 &c. *ad infinitum*.

*Of the best Manner of Proceeding to the Computation of
 more Terms of the said Series, if required.*

ART. 8. If more than four terms of this last series
 are required, it will be necessary to compute the series

$1 + \frac{z}{3e} - \frac{zz}{9ee} + \frac{5z^3}{81e^3} - \frac{10z^4}{243e^4} + \frac{22z^5}{729e^5} - \frac{154z^6}{6561e^6} + \&c.$ (which is $= \sqrt[3]{1 + \frac{z}{e}}$),
 to more than fifteen terms; in order to which it will be
 convenient to express the terms of that series in the

following manner, to wit, $1 + \frac{1}{3} \times \frac{Az}{e} - \frac{2}{6} \times \frac{Bzz}{ee} + \frac{5}{9} \times \frac{Cz^3}{e^3} - \frac{8}{12} \times$
 $\frac{Dz^4}{e^4} + \frac{11}{15} \times \frac{Ez^5}{e^5} - \frac{14}{18} \times \frac{Fz^6}{e^6} + \frac{17}{21} \times \frac{Gz^7}{e^7} - \frac{20}{24} \times \frac{Hz^8}{e^8} + \frac{23}{27} \times \frac{Iz^9}{e^9} - \frac{26}{30} \times \frac{Kz^{10}}{e^{10}} + \frac{29}{33} \times$

$$\begin{aligned} & \frac{Tz^{11}}{e^{11}} - \frac{32}{36} \times \frac{Mz^{12}}{e^{12}} + \frac{35}{39} \times \frac{Nz^{13}}{e^{13}} - \frac{38}{42} \times \frac{Oz^{14}}{e^{14}} + \frac{41}{45} \times \frac{Pz^{15}}{e^{15}} - \frac{44}{48} \times \frac{Qz^{16}}{e^{16}} + \frac{47}{51} \times \\ & \frac{Rz^{17}}{e^{17}} - \frac{50}{54} \times \frac{Sz^{18}}{e^{18}} + \frac{53}{57} \times \frac{Tz^{19}}{e^{19}} - \frac{56}{60} \times \frac{Vz^{20}}{e^{20}} + \frac{59}{63} \times \frac{Wz^{21}}{e^{21}} - \frac{62}{66} \times \frac{Xz^{22}}{e^{22}} + \&c. \text{ or} \\ & A + \frac{Bz}{e} - \frac{Czz}{ee} + \frac{Dz^3}{e^3} - \frac{Ez^4}{e^4} + \frac{Fz^5}{e^5} - \frac{Gz^6}{e^6} + \frac{Hz^7}{e^7} - \frac{Iz^8}{e^8} + \frac{Kz^9}{e^9} - \frac{Lz^{10}}{e^{10}} + \frac{Mz^{11}}{e^{11}} - \frac{Nz^{12}}{e^{12}} + \\ & \frac{Oz^{13}}{e^{13}} - \frac{Pz^{14}}{e^{14}} + \frac{41}{45} \times \frac{Pz^{15}}{e^{15}} - \frac{44}{48} \times \frac{Qz^{16}}{e^{16}} + \frac{47}{51} \times \frac{Rz^{17}}{e^{17}} - \frac{50}{54} \times \frac{Sz^{18}}{e^{18}} + \frac{53}{57} \times \frac{Tz^{19}}{e^{19}} - \frac{56}{60} \times \\ & \frac{Vz^{20}}{e^{20}} + \frac{59}{63} \times \frac{Wz^{21}}{e^{21}} - \frac{62}{66} \times \frac{Xz^{22}}{e^{22}} + \&c; \text{ in which it is evident that} \end{aligned}$$

the generating fractions of the coefficients of the several terms are derived from those that immediately precede them by the continual addition of the number 3 to both their numerators and denominators.

An example of the resolution of a cubick equation by means of the expression $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} + \&c.$ given in Art. 5.

ART. 9. Let it be required to resolve the equation $x^3 - x = \frac{1}{3}$ by means of the said expression.

Here q is $= 1$, and $r = \frac{1}{3}$; and consequently q^3 is $= 1$, and $\frac{q^3}{27} = \frac{1}{27}$, and $\frac{r}{2} = \frac{1}{6}$, and $\frac{rr}{4} = \frac{1}{36}$, which is less than $\frac{1}{27}$, or $\frac{q^3}{27}$. Therefore this equation does not come under CARDAN'S rule, but may be resolved by the expression given in Art. 5, provided that $\frac{rr}{4}$, though less than $\frac{q^3}{27}$, is greater than half $\frac{q^3}{27}$, or than $\frac{q^3}{54}$; which it is, because

it is $=\frac{1}{36}$, whereas $\frac{2^3}{54}$ is equal only to $\frac{1}{54}$, which is less than $\frac{1}{36}$. Therefore the proposed equation $x^3 - x = \frac{1}{3}$ may be resolved by means of the said expression.

ART. 10. Now, since in this case q is $= 1$, and r is $= \frac{1}{3}$, we shall have $\frac{r}{2}$, or e , $= \frac{1}{6}$, and $\frac{q^3}{27} - \frac{rr}{4} (= \frac{1}{27} - \frac{1}{36} = \frac{4}{108} - \frac{3}{108} = \frac{1}{108}) = \frac{1}{36 \times 3}$; that is, zz will be $= \frac{1}{36 \times 3}$, and consequently z will be $= \frac{1}{6\sqrt{3}}$. Therefore $e + z$ will be $= \frac{1}{6} + \frac{1}{6\sqrt{3}} (= \frac{\sqrt{3}}{6\sqrt{3}} + \frac{1}{6\sqrt{3}} = \frac{3}{6 \times 3} + \frac{\sqrt{3}}{6 \times 3} = \frac{3 + \sqrt{3}}{6 \times 3} = \frac{3 + \sqrt{3}}{18} = \frac{3 + 1.732,050,8}{18} = \frac{4.732,050,8}{18}) = .262,891,71$; and $e - z$ will be $= \frac{1}{6} - \frac{1}{6\sqrt{3}} (= \frac{3 - \sqrt{3}}{18} = \frac{3 - 1.732,050,8}{18} = \frac{1.267,949,2}{18}) = .070,441,62$. Therefore the cube-root of $e + z$ is $= \sqrt[3]{.262,891,71} = .640,607,91$; and the cube-root of $e - z$ is $= \sqrt[3]{.070,441,62} = .412,993,40$; and consequently $\sqrt[3]{e + z} + \sqrt[3]{e - z}$ is $= .640,607,91 + .412,993,40 = 1.053,601,31$.

ART. 11. It remains that we compute the infinite series $\frac{Cz^2}{e^2} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{16}}{e^{16}} + \&c.$ and extract the cube-root of e , and then multiply the said series into 4 times the said cube-root.

Now the cube-root of e is in this case $= \sqrt[3]{\frac{1}{6}} = \frac{1}{\sqrt[3]{6}} = \frac{1}{1.817,121}$; and consequently $4\sqrt[3]{e}$ is $= \frac{4}{1.817,121}$.

And,

And, since zz is $= \frac{1}{36 \times 3}$, and ee is $= \frac{1}{36}$, it follows that $\frac{zz}{ee}$ will be $= \frac{1}{3}$. Therefore $\frac{z^4}{e^4}$ will be $= \frac{1}{9}$, and $\frac{z^6}{e^6} = \frac{1}{27}$, and $\frac{z^{10}}{e^{10}} (= \frac{z^6}{e^6} \times \frac{z^4}{e^4} = \frac{1}{27} \times \frac{1}{9}) = \frac{1}{243}$, and $\frac{z^{14}}{e^{14}} (= \frac{z^{10}}{e^{10}} \times \frac{z^4}{e^4} = \frac{1}{243} \times \frac{1}{9}) = \frac{1}{2187}$. Therefore the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \&c.$ will

in this case be $= \frac{C}{3} + \frac{G}{27} + \frac{L}{243} + \frac{P}{2187} + \&c. = \frac{.111,111,111, \&c.}{3} + \frac{.023,472,031, \&c.}{27} + \frac{.011,690,017, \&c.}{243} + \frac{.007,414,542, \&c.}{2187} + \&c. = .037,037,037, \&c. + .000,869,334, \&c. + .000,048,107, \&c. + .000,003,390, \&c. + \&c. = .037,957,868, \&c.$

Therefore $4\sqrt[3]{e} \times$ the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \&c.$ is equal to $\frac{4}{1.817,121} \times .037,957,868, \&c. = \frac{.151,831,472, \&c.}{1.817,121} = .083,556,06.$

Consequently the whole expression $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \&c.$ is $= 1.053,601,31 + .083,556,06 = 1.137,157,37$; that is, the root of the proposed equation $x^3 - x = \frac{1}{3}$ is $= 1.137,157,37$. Q. E. I.

ART. 12. This value of the root of the equation $x^3 - x = \frac{1}{3}$ is exact to six places of figures, the more accurate value of it being 1.137,158,164. We may therefore conclude that the expression here made use of to determine the value of x , to wit, $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$

$4\sqrt[3]{e} \times$ the infinite series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \&c.$ is somewhat preferable, with respect to the practical resolution of these equations, to the other expression of its value given in the former paper in the Philosophical Transactions, to wit, $\sqrt[3]{e} \times$ the infinite series $2 + \frac{2zz}{9ce} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} - \&c.$ For it appeared in Art. 42 of that paper, page 941, that the value of the root of this same equation $x^3 - x = \frac{1}{3}$ obtained by computing four terms of the series $2 + \frac{2zz}{9ce} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} - \&c.$ was 1.137,33; which is true only to four places of figures; whereas by computing the same number of terms of the series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \&c.$ we have just now obtained a value of the same root, to wit, the number 1.137,157,37, which is exact to six places of figures. This is agreeable to what was observed above in Art. 6.

A Summary of the Conclusions obtained in this Paper and the former Paper to which it is an Appendix.

ART. 13. I will now conclude this paper by setting down, in as concise a manner as I can, the several conclusions that have been obtained in this and the above-mentioned paper in the Philosophical Transactions for the

the year 1778, Number XLII. page 902, &c. concerning the root of the cubick equation $x^3 - qx = r$, that the whole may be seen together at one view.

ART. 14. If $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, and e be put $= \frac{r}{2}$, and $ss = \frac{rr}{4} - \frac{q^3}{27}$, it is shewn in Art. 5, of the said former paper, that the root of the equation $x^3 - qx = r$ will be $= \sqrt[3]{\left[\frac{r}{2} + \sqrt{\left[\frac{rr}{4} - \frac{q^3}{27}\right]}\right]} \frac{+q}{3\sqrt[3]{\left[\frac{r}{2} + \sqrt{\left[\frac{rr}{4} - \frac{q^3}{27}\right]}\right]}}$, or $\sqrt[3]{e+s} + \frac{q}{3\sqrt[3]{e+s}}$.

ART. 15. And it is shewn in Art. 9. of the said paper, that the root of the said equation will in that case be also equal to $\sqrt[3]{\left[\frac{r}{2} - \sqrt{\left[\frac{rr}{4} - \frac{q^3}{27}\right]}\right]} \frac{q}{3\sqrt[3]{\left[\frac{r}{2} - \sqrt{\left[\frac{rr}{4} - \frac{q^3}{27}\right]}\right]}}$, or $\sqrt[3]{e-s} + \frac{q}{3\sqrt[3]{e-s}}$.

ART. 16. And it is shewn in Art. 11, of the said paper, page 915, that the root of the said equation will in that case be also equal to $\sqrt[3]{\left[\frac{r}{2} + \sqrt{\left[\frac{rr}{4} - \frac{q^3}{27}\right]}\right]} + \sqrt[3]{\left[\frac{r}{2} - \sqrt{\left[\frac{rr}{4} - \frac{q^3}{27}\right]}\right]}$, or $\sqrt[3]{e+s} + \sqrt[3]{e-s}$.

ART. 17. And it is shewn in Art. 23, of the said paper, page 923, that the root of the said equation will in that case be also equal to $\sqrt[3]{e} \times$ the infinite series

$$2 - \frac{2ss}{9e^2} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \frac{1870s^8}{59049e^8} - \frac{.111,826s^{10}}{4,782,969e^{10}} - \frac{2,358,512s^{12}}{129,140,163e^{12}} - \frac{120,646,960s^{14}}{8,135,830,269e^{14}}$$

$\frac{120,646,960s^{14}}{8,135,830,269e^{14}} - \&c.$ or (if we put the capital letters A, B, C, D, E, F, G, H, &c. for the several numeral coefficients 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{5}{81}$, $\frac{10}{243}$, $\frac{22}{729}$, $\frac{154}{6561}$, $\frac{2618}{137,781}$, &c. of the terms of the series $1 + \frac{s}{3e} - \frac{ss}{9e^2} + \frac{5s^3}{81e^3} - \frac{10s^4}{243e^4} + \frac{22s^5}{729e^5} - \frac{154s^6}{6561e^6} + \frac{2618s^7}{137,781e^7} - \&c.$ which is equal to the cube-root of the binomial quantity $1 + \frac{s}{e}$), equal to $\sqrt[3]{e} \times$ the infinite series $2A - \frac{2Cs}{ee} - \frac{2Es^4}{e^4} - \frac{2Gs^6}{e^6} - \frac{2Is^8}{e^8} - \frac{2Ls^{10}}{e^{10}} - \frac{2Ns^{12}}{e^{12}} - \frac{2Ps^{14}}{e^{14}} - \&c.$; in which series all the terms after the first term $2A$, or 2 , are marked with the sign $-$, or are to be subtracted from the said first term.

ART. 18. And, if $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, but greater than its half, or $\frac{q^3}{54}$, and e be put, as before, $= \frac{r}{2}$, and $xx = \frac{q^3}{27} - \frac{rr}{4}$, it is shewn in Art. 31, 32, 33, 34, 35, of the said paper, pages 927—936, that the root of the equation $x^3 - qx = r$ will be equal to $\sqrt[3]{e} \times$ the infinite series $2 + \frac{2xz}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} - \frac{1870z^8}{59049e^8} + \frac{111,826z^{10}}{4,782,969e^{10}} - \frac{2,358,512z^{12}}{129,140,163e^{12}} + \frac{120,646,960z^{14}}{8,135,830,269e^{14}} - \&c.$ or $\sqrt[3]{e} \times$ the infinite series $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} - \frac{2Iz^8}{e^8} + \frac{2Lz^{10}}{e^{10}} - \frac{2Nz^{12}}{e^{12}} + \frac{2Pz^{14}}{e^{14}} - \frac{2Rz^{16}}{e^{16}} + \frac{2Tz^{18}}{e^{18}} - \&c.$; in which series the capital letters A, C, E, G, I, L, N, P, R, T, &c. denote the same numeral coefficients 1, $\frac{1}{9}$, $\frac{10}{243}$, $\frac{154}{6561}$, &c. as in the last Article, and all the terms that involve the

odd powers of the fraction $\frac{zz}{ee}$, to wit, $\frac{zz}{ee}$ itself, $\left[\frac{zz}{ee}\right]^3$, or $\frac{z^6}{e^6}$, $\left[\frac{zz}{ee}\right]^5$, or $\frac{z^{10}}{e^{10}}$, $\left[\frac{zz}{ee}\right]^7$, or $\frac{z^{14}}{e^{14}}$, $\left[\frac{zz}{ee}\right]^9$, or $\frac{z^{18}}{e^{18}}$, &c. are marked with the sign +, or are to be added to the first term 2A, or 2, and all the other terms are marked with the sign -, or are to be subtracted from the former.

These are the conclusions obtained in the said former paper, which is printed in the Philosophical Transactions for the year 1778, Number XLII. pages 902—949. The conclusions obtained in this paper are as follows.

ART. 19. If $\frac{rr}{4}$ is less than $\frac{r^3}{27}$, but greater than its half, or $\frac{r^3}{54}$, and e be put, as before, $= \frac{r}{2}$, and $zz = \frac{r^3}{27} - \frac{rr}{4}$, it is shewn in the present paper, Art. 5 and 7, that the root of the equation $x^3 - qx = r$ will be equal to the mixed expression $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the infinite series $\frac{zz}{9e} + \frac{154z^6}{6561e^6} + \frac{55913z^{10}}{41782,969e^{10}} + \frac{60,323,480z^{14}}{8,135,830,269e^{14}} + \&c.$ or $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the infinite series .III, III, III, &c. $\times \frac{zz}{ee} + .023,472,031, \&c. \times \frac{z^6}{e^6} + .011,690,017, \&c. \times \frac{z^{10}}{e^{10}} + .007,414,542, \&c. \times \frac{z^{14}}{e^{14}} + \&c.$ or $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the infinite series $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \&c;$ in which series the capital letters C, G, L, P, &c. denote the

same numeral coefficients $\frac{1}{9}$, $\frac{154}{6561}$, $\frac{55913}{4,782,969}$, $\frac{60,323,480}{8,135,830,269}$, &c. as they denoted in the two last Articles, to wit, the coefficients of $\frac{zx}{e^2}$, $\frac{z^6}{e^6}$, $\frac{z^{10}}{e^{10}}$, $\frac{z^{14}}{e^{14}}$, $\frac{z^{18}}{e^{18}}$, &c. or of the odd powers of $\frac{zx}{e^2}$ in the series which is equal to $\sqrt[3]{1 + \frac{z}{e}}$. And it is also shewn in Art. 6 and 12 of this paper, that this expression, computed to a given number of terms, gives the value of x somewhat more exactly than the former expression, $\sqrt[3]{e} \times$ the infinite series $2 + \frac{2zx}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} -$ &c. if computed only to the same number of terms.

ART. 20. As to the second branch of the second case of the equation $x^3 - qx = r$, or that in which $\frac{rr}{4}$ is less than half $\frac{q^3}{27}$, or than $\frac{q^3}{54}$, I do not know any method of extending CARDAN'S rule to it. But I have been informed by my learned and ingenious friend Dr. CHARLES HUTTON, Professer of Mathematicks in the Royal Academy at Woolwich, that he has discovered such a method: and I hope he will soon communicate it to this learned Society.

